Limit Theorems for the Tagged Particle in in Exclusion Processes on Regular Trees

Dayue CHEN

Peking University

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Outline

1. The model
2. new results
3. some remarks
4. sketches of proofs

The exclusion process is an interacting particle system.

Underlying space S is a graph (V, E).

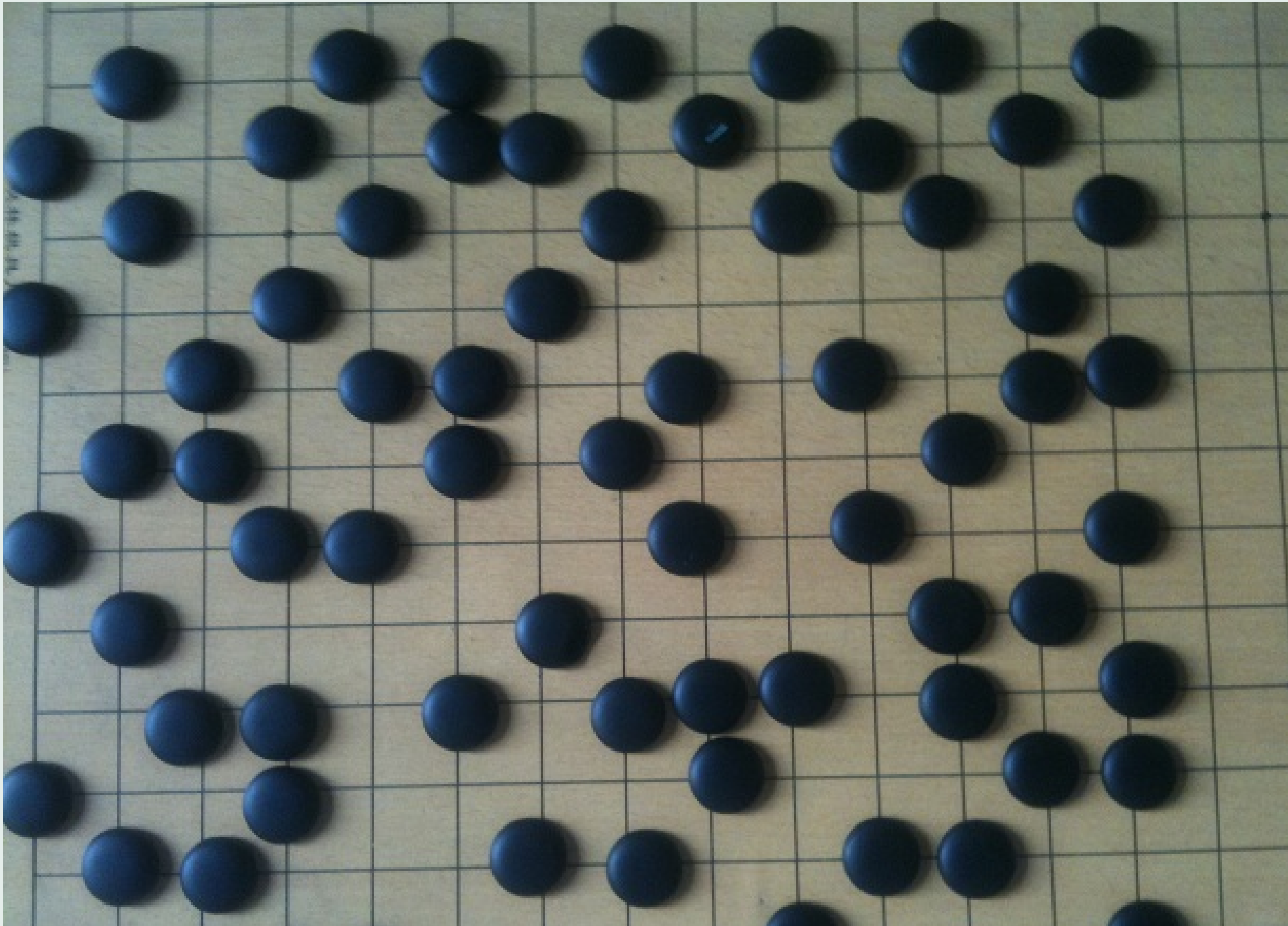
The default choice is the lattice Z^d

configuration η = a point of $\{0, 1\}^V$.

$\eta = \{\eta(x); x \in V\}$.

$\eta(x)$ = 1, a particle at x

$\eta(x)$=0, site x is vacant.



Transition mechanisms of particles.

1. at most one particle in every site.

2. A particle at x waits for an exponential time and attempts to jump to another site y with probability $p(x, y)$.

3. If y is vacant, particle moves to y; if y is occupied, then particle stays in x and the attempt is suspended.

$p(x, y)$ is the transition probability of a Markov chain on S.

$\eta \longrightarrow \eta^{xy}$ at rate $p(x, y)$

$$\Omega f(\eta) = \sum_{x,y \in S} \eta(x)(1 - \eta(y))p(x, y)[f(\eta^{xy}) - f(\eta)].$$

T.M Liggett, *Interacting Particle Systems*. Grundlehren der Mathematischen Wissenschaften, **276**. Springer-Verlag, Berlin, 1985

T.M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*. Grundlehren der Mathematischen Wissenschaften, **324**. Springer-Verlag, Berlin, 1999.

$\{p(x,y)\}$ is the transition probability of a Markov chain on S.

Extra assumptions on $p(x,y)$.

E.g. $p(x,y) = 1/d_x$ if $|x-y| = 1$ and $p(x,y) = 0$ if $|x-y| \neq 1$.

There is no birth and death, the density of particles is preserved.

The Bernoulli product measure μ_ρ is invariant (and ergodic).

Not easy to identify all invariant measures. symmetric or Z^1 nearest neighbor, or Z^1 mean zero.

From now on, simple exclusion

initial measure = the Bernoulli product measure μ_ρ.

(although the conclusions could be valid in a more general setting.)

Simple exclusion process on a tree.

T_d = regular tree of degree d.

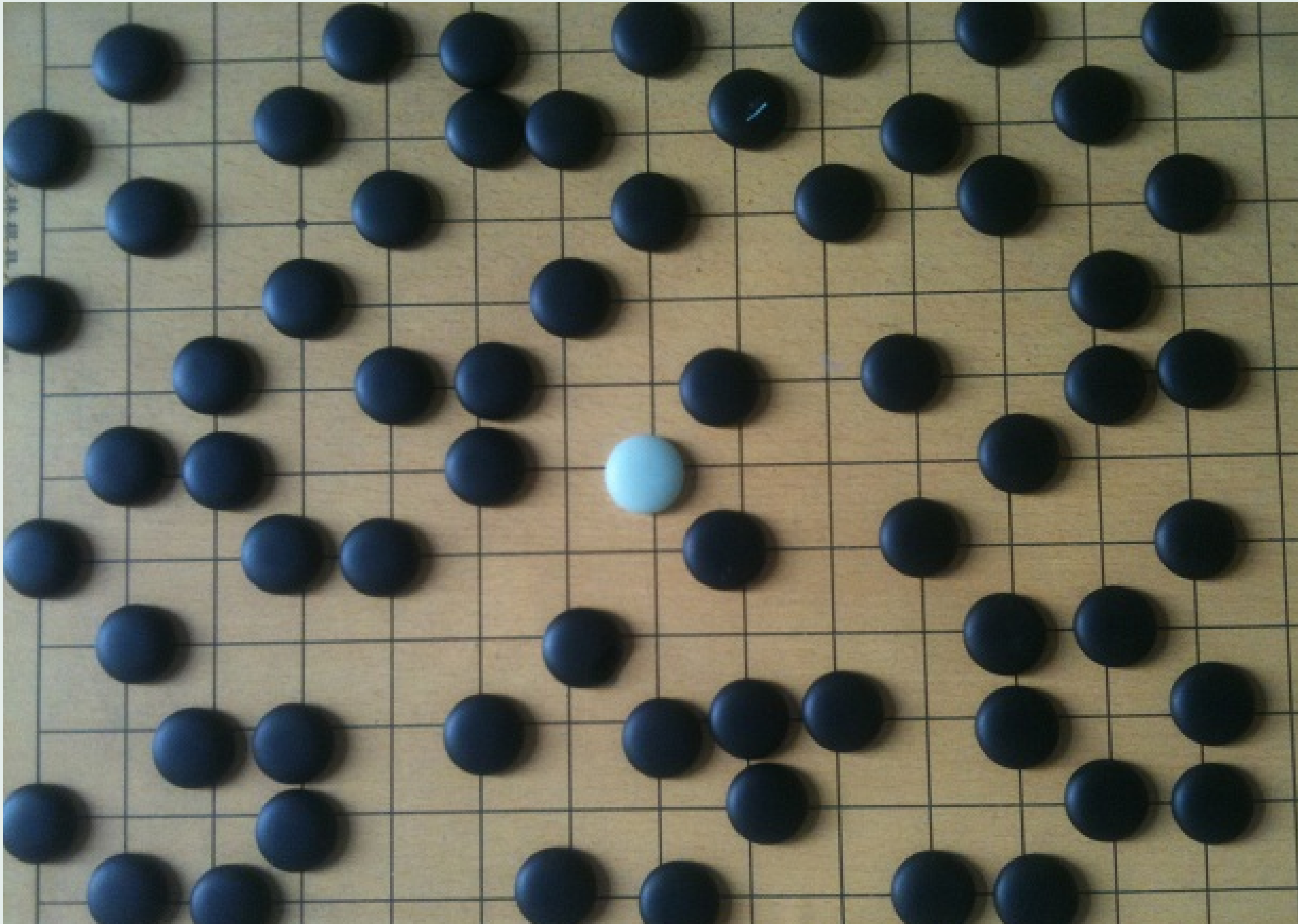
Simple random walk on T_d.

Fix a site as the root and the walker is at the root initially.

The walk waits for an exponential time with parameter 1, and moves to a neighboring site with probability $1/d$ when the clock rings.

Mark a particle (called the tagged particle).

Goal: to study the motion $X(t)$ of the tagged particle.



$X(t)$ behaves very much like a random walk on S,

except some suspensions due to collision with other particles.

$\xi_t(x) = \eta(X_t + x)$

$\xi_t = \{\xi_t(x), x \in T_d\}$, the environment seen from X_t.

$$(\eta_t, X_t) \longleftrightarrow (X_t, \xi_t)$$

II. New Results

1. Ergodicity of $\boldsymbol{\mu}^*_{\boldsymbol{\rho}}$,

2. LLN of $|\boldsymbol{X_t}|$, where $|\boldsymbol{x}| = \text{dist}(\ \boldsymbol{x}, \boldsymbol{o})$.

3. CLT of $|\boldsymbol{X_t}|$.

μ_ρ = the Bernoulli product measure = initial distribution.

μ^*_ρ = μ_ρ conditioned on there is a particle at the root.

Proposition 1. For $d \geq 3$, the measure ν^*_ρ is invariant and ergodic for $(\xi_t)_{t\geq 0}$ and all $\rho \in (0,1)$.

Theorem 2. For $d \geq 2$, let $(\eta_t)_{t \geq 0}$ on T_d have initial distribution ν^*_ρ for some $\rho \in [0, 1]$. Then the position of the tagged particle $\{X_t; t \geq 0\}$ satisfies a law of large numbers:

$$\lim_{t \to \infty} \frac{|X_t|}{t} = (1 - \rho)(d - 2) \sum_{i \in \mathbb{N}_0} ip(i) =: v$$

P_{ν_ρ}-almost surely. In particular, speed = $(1 - \rho)\frac{d-2}{d}$ in the case of the simple exclusion process on T_d.

Theorem 3. For $d \geq 3$ and $\rho \in [0, 1)$, the tagged particle $(X_t)_{t \geq 0}$ on T_d satisfies

$$\frac{|X_t| - tv}{\sqrt{t}} \xrightarrow{\text{d}} \mathcal{N}(0, \sigma^2)$$

for some $\sigma = \sigma(d, \rho, p(.)) \in (0, \infty)$ and v from Theorem 1.

III Earlier Works on $\mathbb{Z}^d$.

$$\lim_{t\to\infty}\frac{X(t)}{t}=(1-\rho)\sum_y y p(0,y) \qquad a.s.$$

was first verified in two cases:

1) $S=Z^1$ and $p(x,x+1)=1$ (totally asymmetric).

2) $S=Z^1$ and $p(x,x+1)=p(x,x-1)=1/2$

Key step: to verify that the environment viewed from the tagged particle is stationary and ergodic.

Ergodicity of μ_ρ can not be inherited automatically when μ_ρ is conditioned on $\eta(0) = 1$.

Assuming translational invariance $p(x, y) = p(0, y - x)$ for all x, y, this was done by E. Saada in the following cases:

1) Z^d, $d \geq 2$,

2) Z^1, $p(x, x+1) + p(x, x-1) < 1$.

and by P.A. Ferrari

3) Z^1, $p(x, x+1) + p(x, x-1) = 1$.

CLT (Kipnis 85, Kipnis & Varadhan 85).

$$Z_t = \frac{X_t - EX_t}{\sqrt{t}}$$

is asymptotically normal if

1) $S = Z^1$, $p(x, x+1) + p(x, x-1) = 1$; or

2) $S = Z^d$, $p(x, y) = p(y, x) = p(0, y - x)$, irreducibility of the random walk and $\sum_x |x|^2 p(0, x) < \infty$.

But both excludes the case that $S = Z^1$, $p(x, x+1) = p(x, x-1) = 1/2$.

3) $S = Z^d$, $\sum_y y p(0, y) = 0$. (Varadhan 1995)

4) $S = Z^d$, $d \geq 3$, $\sum_y y p(0, y) \neq 0$. (Sethuraman, Varadhan and Yau 1999)

However, if $S = Z^1$ and $p(x, x+1) = p(x, x-1) = 1/2$, $X(t)$ is not a RW with a certain rate of suspension. It is subdiffusive.

Theorem. If the initial distribution is the Bernoulli product measure μ_ρ conditioned on $\eta(0) = 1$. Then $X_t/t^{1/4}$ converges in distribution to the normal law with mean zero and variance $\sqrt{2/\pi}(1-\rho)/\rho$. Furthermore

$$\lim_t \frac{var(X_t)}{\sqrt{t}} = \sqrt{\frac{2}{\pi}}\frac{1-\rho}{\rho}.$$

Richard Arratia, *The motion of a tagged particle in the simple symmetric exclusion system on* $\mathbb{Z}$, Ann. Probab. **11** (1983), no. 2, 362–373.

Sketches of Proof

Theorem 1. Ergodicity of the environment viewed from the tagged particle, following the idea of E. Saada.

Theorem 2. Linear speed of the tagged particle, following the idea of RW on tree by Russell Ryons. A new distance is introduced

Theorem 3. Central Limit Theorem of the tagged particle, following the idea of Sethuraman, Varadhan and Yau (1999)

Proof of Proposition 1.

reducing ergodicity of ν_ρ^* from that of ν_ρ.

$S = \{0,1\}^V$, $S^* = \{\eta \in S, \eta(0) = 1\}$.

If ν_ρ^* is not ergodic, find an invariant $A \subset S^*$, $0 < \nu_\rho^*(A) < 1$,

$$\overline{A} = \{\tau_x \eta, \eta \in A, x \in V\}.$$

Then $\nu_\rho(\overline{A}) > 0$, so $\nu_\rho(\overline{A}) = 1$.

$\boldsymbol{B = S \setminus A}$ is invariant, $\boldsymbol{0 < \nu_\rho^*(B) < 1}$, $\boldsymbol{\overline{B} = \{\tau_x\eta, \eta \in B, x \in V\}}$.

Then $\boldsymbol{\nu_\rho(\overline{B}) > 0}$, $\boldsymbol{\nu_\rho(\overline{B}) = 1}$.

$\boldsymbol{\overline{A}}$ and $\boldsymbol{\overline{B}}$ are almost the same. Therefore one can pick a point from $\boldsymbol{\overline{A} \cap \overline{B}}$.

$\exists \boldsymbol{x, w} \in \boldsymbol{V}$, $\boldsymbol{\tau_x\eta} \in \boldsymbol{A}$ and $\boldsymbol{\tau_w\eta} \in \boldsymbol{B}$.

We will argue that $\boldsymbol{\tau_w\eta^{xy}} \in \boldsymbol{A}$ and $\boldsymbol{\tau_w\eta^{xy}} \in \boldsymbol{B}$, a contradiction.

For this we can find integers n, m, l and sites

$y, z; x_1, x_2, \cdots, x_n; y_1, y_2, \ldots, y_m; \;\; z_1, z_2, \ldots, z_l$

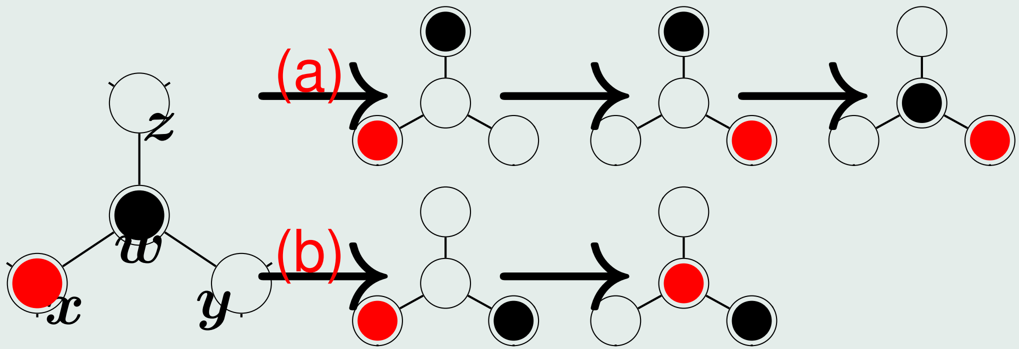
$\eta(y) = \eta(z) = \eta(x_1) = \ldots = \eta(x_n) = 0$

x, y, z are located in pairwise different branches w. r. t. w in T^d

w is connected to x via the path $x_1 \sim x_2 \sim \cdots \sim x_n$,

connected to y via the path $y_1 \sim y_2 \sim \cdots \sim y_m$ and

connected to z via the path $z_1 \sim z_2 \sim \cdots \sim z_l$.


z
w
x
y
(a)
(b)

Extend to Galton-Watson trees for the speed existence.

dimension drop?

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Thank You

E-mail: dayue@pku.edu.cn